Homework in Representation theory

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Question 1

(i) Suppose W is not irreducible. Then it contains a invariant non-trivial subspace V which must have dimension 1 and so be spanned by some non-zero vector $u = (u_1, u_2, u_3) \in \mathbb{C}^3$ such that $u_1 + u_2 + u_3 = 0$. Since V is invariant under the cyclic shift (123) there must exist a number k such that $u_i = ku_{i+1} = k^2 u_{i+2} = k^3 u_i$ for each i mod 3. Since $u \neq 0$, we have $k^3 = 1$. Hence $u = u_1(1, k, k^2)$ (u_1 must be non-zero, otherwise u is zero) so $u_1(1 + k + k^2) = 0$ which means that $u = (1, \omega, \omega^2)$ where ω is one of the two third roots of unity different from 1. Since V is invariant there must also be some t such that $(1, \omega, \omega^2) = t(\omega, 1, \omega^2)$. However the vectors $(1, \omega, \omega^2)$ and $(\omega, 1, \omega^2)$ are not parallel, a contradiction. We conclude that W is irreducible.

(ii) Since we know there are as many irreps as conjugacy classes, and there are three partitions of 3 ($3 = 1 + 1 + 1$, $3 = 2 + 1$, $3 = 3$) corresponding to the cycle types in S_3, the given list is an exhausting one. The elements in the given list are non-isomorphic since W has different dimension from the other two and the trivial representation is a constant, unlike the alternating representation.

Question 2

I'm not sure I understand this question. Exactly this is done in a direct manner in Fulton and Harris' book, and the multiplicity of any irrep V is $\dim V$ by character theory.

Question 3

(i) An arbitrary element of $\bigwedge^2 V$ can be written $\sum_i \wedge_{j=1}^{n} (u_{ij} \oplus w_{ij})$ where $u_{ij} \in U$, $w_{ij} \in W$.

Expanding this formally and grouping all terms coming from U to the left, we get a sum of terms of the following form $\wedge_{j=1}^{i} u'_j \wedge_{j=1}^{n-p} w'_j$ for varying i. But this is precisely the description of a term of the RHS, giving us an isomorphism.

(ii) I just compute the dimension of the two sides, without giving a bijective proof. Let $\dim U = r, \dim W = s$, so that $\dim V = rs$. We compute $\dim(RHS) = \binom{r+1}{2}\binom{s}{2} + \binom{r}{2}\binom{s+1}{2} = \frac{1}{4}\left((r+1)rs(s-1) + r(r-1)(s+1)s\right) = \frac{1}{2}\left(rs(rs-1)\right) = \dim(LHS)$.

Question 4

By definition, $\chi_{S^2V}(g)$ is the trace of the mapping $g \otimes g$, which is the sum of its eigenvalues. So we need to examine the eigenvalues of $g \otimes g$. By finiteness of G and answer 11, g is diagonalizable and has eigenvectors v_i with eigenvalues λ_i spanning V.

Claim: the vectors $\{v_i \otimes v_j\}_{i \leq j}$ are linearly independent. Suppose not, then there is a relation $\sum_{i \leq j} c_{ij} v_i \otimes v_j = 0$. Since $\{v_j\}$ form a basis, for each j we have $\sum_{i \leq j} c_{ij} v_i = 0$. Again, since $\{v_i\}$ are linearly independent we get $c_{ij} = 0$ for all i, j.

Since the given set of vectors has size $\binom{n+1}{2}$ which also is the dimension of S^2V, they form an eigenbasis of S^2V. Moreover the eigenvalue of $v_i \otimes v_j$ is $\lambda_i\lambda_j$ since $g(v_i \otimes v_j) = gv_i \otimes gv_j = \lambda_i\lambda_j v_i \otimes v_j$. So we need to express the sum $\sum_{i \leq j} \lambda_i\lambda_j$ in terms of $\left(\sum_i \lambda_i\right)^2$ and $\sum_i \lambda_i^2$, the latter being the sum of the eigenvalues of g^2. This is easy, we have

$$\sum_{i \leq j} \lambda_i\lambda_j = \frac{1}{2}\left(\left(\sum_i \lambda_i\right)^2 + \sum_i \lambda_i^2\right),$$

which is what we wanted to prove.

Question 5

I follow the solution given in Fulton-Harris, proving some assertions more carefully.

Let χ be the character of ϱ and φ the character of any (irreducible) representation. We need to prove that the sequence $a_i := (\varphi, \varrho^i)$, $i \geq 0$ is not identically zero, as a_i is the multiplicity of representation corresponding to φ in $V^{\otimes i}$.

The generating function of this sequence is $\sum_{i=0}^{\infty} a_i t^i = \sum_{i=0}^{\infty} \frac{1}{\#G} \sum_C \#C\overline{\varphi(C)}\chi(C)^n t^n = \frac{1}{\#G} \sum_C \frac{\#C\overline{\varphi(C)}}{1-\chi(C)t}$, and we need to prove that it is not identically zero. We do this by showing that the coefficient in front of $\frac{1}{1-t\dim V}$ is not zero. This follows from two claims: (i) the only term in the sum contributing to this coefficient is the one where C contains the identity element (ii) that term is non-zero.

Proof of claim (i): $\chi(C) = \mathrm{tr}\varrho g$, the sum of the eigenvalues of ϱg, where $g \in C$. Since $g^{\#G} = 1$, the eigenvalues $\lambda_1, \ldots, \lambda_n$, where $n = \dim V$, of ϱg are roots of unity. Consequently $|\lambda_1 + \cdots + \lambda_n| \leq |\lambda_1| + \cdots + |\lambda_n| = \dim V$ with equality iff $\lambda_i = 1$ for all i, by the triangle inequality. Hence ϱg is the identity, since $\varrho g - Id$ has all eigenvalues equal to 0 and is semisimple, hence zero. Since ϱ is injective, $g = 1$. Proof of claim (ii): $\overline{\varphi(C)} =' \dim' \varphi > 0$

Question 6

I follow the solution given in Fulton-Harris, proving some assertions more carefully.

For any conjugacy class C of G, define the G-linear map $\varphi_C = \sum_{g \in C} g$. By Schur's lemma, there is a complex number λ_C such that φ_C is the identity times λ_C. Hence $\mathrm{tr}\varphi = \lambda_C \dim V$. Also, $\mathrm{tr}\varphi = \sum_{g \in C} \mathrm{tr} g = \#C\chi_V(C)$ by definition of χ_V.

We now show that $\sum_{g \in C} g$, as C varies of conjugacy classes of G, generate the center of $\mathbb{Z}[G]$. That $z \in Z(\mathbb{Z}[G])$ means that $z = \sum_{g \in G} c_g g$ and that $zg = gz$ for each $g \in G$. Hence $c_{ghg^{-1}}$ is equal to c_h for any $g, h \in G$, which means that c_g only depends on the conjugacy class of g, and so is a $\mathbb{Z}$-linear combination of the elements $\sum_{g \in C} g$. So $Z(\mathbb{Z}[G])$ is a finitely generated abelian group. In such a group, for any element x, the chain $\{\{1, x, \ldots, x^r\}\}_r$ stabilizes. This means some power of x is equal to a linear combination of lower powers of x, or x is an algebraic integer (root of monic polynomial with integer coefficients). Thus the λ_C are all algebraic integers.

Since $\#G = \sum_C \#C\overline{\chi_V(C)}\chi_V(C) = \dim V \sum_C \lambda_C \overline{\chi_V(C)}$ (equivalent to $(\chi_V, \chi_V) = 1$). We see that $\#G/\dim V$ too is an algebraic integer. However $\#G$ and $\dim V$ are both positive integers hence $\#G/\dim V$ is a positive rational algebraic integer, or simply put, a positive integer.

Question 7 Since $g^2 = 1$, the eigenvalues of g square to 1 and hence are 1 or -1. Let n_+ be the number of $+1$s and n_- the number of -1s. We have $n_+ + n_- = \dim V = \chi_V(1)$ and consequently $\chi_V(g) = n_+ - n_- = \chi_V(1) - 2n_-$, and n_- clearly is a nonnegative integer.

Question 8

The character table will be indexed by conjugacy classes and irreps. We first find the conjugacy classes. They are the conjugacy classes of S_4 split up further according to parity (which results in some empty classes, which we omit).

The nonempty classes are represented by 1 (identity), (123), (132), and (12)(34). This gives us the number of elements in each conjugacy class. So the top of the character table will look as follows

	1	4	4	3
A_4	1	(123)	(132)	(12)(34)

We should fill in four rows corresponding to the four (= #conjugacy classes) irreps of A_4. As always, we have the trivial representation $U \mid 1 \quad 1 \quad 1 \quad 1$.

Since A_4 is a subgroup of S_4 we could examine the representation given by permuting coordinates in $\mathbb{C}^4$, call it V_1. The value of the character of a permutation now becomes the number of fixed points it has, its 'row' is $V_1 \mid 4 \quad 1 \quad 1 \quad 0$. We know that this representation certainly is not irreducible as it is the sum of the trivial representation and some representation V_2. This representation V_2 will have row $V_2 \mid 3 \quad 0 \quad 0 \quad -1$. Since the inner product of this row with itself is 1, we find that V_2 is an irrep.

Since A_4 has a cyclic (hence abelian) subgroup of order 3 (but none of order 4) it is natural to consider it. The irreps. in for this group will be one-dimensional so we use representation and character interchangeably in what follows.

The character χ of any non-trivial representation V_3 of this group assigns ω or ω^2 to the cycle (123), where ω is a third root of unity. Moreover, since (123) and (132) are inverses, we have $\chi((132)) = \overline{\chi((123))}$. Finally, since $(123) \cdot (12)(34) = (134)$, the latter belonging to the conjugacy class of (123), we have $\chi((123))\chi((12)(34)) = \chi((123))$, so $\chi((12)(34)) = 1$. The two choices for $\chi((123))$ gives us the last two rows.

	1	4	4	3
A_4	1	(123)	(132)	(12)(34)
U	1	1	1	1
V_2	3	0	0	-1
$V_{3\omega}$	1	ω	ω^2	1
$V_{3\omega^2}$	1	ω^2	ω	1

Since this represents a square matrix with orthonormal rows, we have found all the irreps.

Question 9 A one-dimensional representation is the same thing a morphism from G to $\mathbb{C}^*$, the latter being commutative. Since every such morphism factors through G/G', we may instead count the number of representations of G/G'.

Since G/G' is commutative, all the irreps. have dimension 1. Since the sum of their squares should be $\#G/G' = [G:G']$ on the one hand and $\#(irreps)\cdot 1$ on the other hand, the number of such representations is $[G:G']$.

Question 10

So G is the dihedral group of order $2p$.

(i) This follows from answer 9 since G' in this case are(as is well known) the rotations, which have index 2 in G.

(ii) This implies and is consistent with the definition

$$\rho_m(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\rho_m(b) = \begin{pmatrix} \varepsilon^m & 0 \\ 0 & \varepsilon^m \end{pmatrix}$$

To show that ρ_m is a representation it is enough to calculate the products $\rho_m(a)^2$, $\rho_m(b)^p$, $\rho_m(a)\rho_m(b)\rho_m(a)\rho_m(b)$ and verify that they are equal to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. I did that.

We now check that they are irreducible. If ρ_m is not irreducible there is some invariant 1-dimensional subspace spanned by some $u = (u_1, u_2) \in \mathbb{C}^2$. Since $\rho_m(a)u$ is parallel to u we have WLOG $u = (1,1)$ or $(1,-1)$. If $u = (1,1)$ then $\rho_m(b)u$ being parallel to u is the same thing as $\varepsilon^2 = 1$ which contradicts the choice of p. If $u = (1,-1)$ we similarly obtain $\varepsilon^2 = 1$. Thus ρ_m is irreducible.

We now show the irreps are pariwise nonisomorphic. Suppose ρ_m and $\rho_{m'}$ are isomorphic. We need to show that $m = m'$.

By hypothesis, we have the following equalities for the characters: $\chi_{\rho_m}(b) = \varepsilon + \varepsilon = \vartheta + \vartheta = \chi_{\rho_{m'}}(b)$, where $\varepsilon = \exp(2\pi i m/p)$ and $\vartheta = \exp(2\pi i m'/p)$. Since $z + \frac{1}{z} = w + \frac{1}{w}$ (being equivalent to $(z-w)(zw-1) = 0$ if $z, w \neq 0$) iff $z = w$ or $z = \frac{1}{w}$ we have $\varepsilon = \vartheta$ or $\varepsilon = \vartheta^{-1}$.

Suppose $\varepsilon = \vartheta^{-1}$. Then $m = -m'$ but this contradicts the choice of m and m' being between 1 and $(p-1)/2$. If $\varepsilon = \vartheta$ then $m = m'$ and we are done.

Question 11 Suppose A is not diagonalizable. Then there is a block J ($J = \begin{pmatrix} 1 & 1 & 0 & \dots \\ 0 & 1 & 1 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & & & \end{pmatrix}$) in the Jordan form of A of size at least 2. If B is any matrix of size at least 2×2, let $TL_{2,2}(B)$ be the top left 2×2 submatrix of B. It is easy to see that $TL_{2,2}(J^k) = TL_{2,2}(J)^k$ so it suffices to show that no positive power of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is the identity. The k:th power of that matrix is $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, so we are done.

Question 12 I use the construction of $V := Ind_G^H U$ given in Fulton-Harris, 3.3.

Let $g \in C$, and u_σ a vector spanning U^σ for each left coset σ of H in G. Furthermore let $g_\sigma \in G$ be a representative of the left coset σ for each σ. By construction of V, $gg_\sigma = g_\tau$, where τ is the left coset to which gg_σ belongs. Writing g in the basis $\{u_\sigma\}$ we find that the trace of g is the same as the number of σ such that $gg_\sigma = g_\sigma h$ for some $h \in H$, that is, the number of σ such that $g_\sigma^{-1} g g_\sigma \in H$.

Since this does not depend on g we have

$$\chi_V(C) = \frac{1}{\#C} \sum_{g \in C} \sum_{\sigma} I(g_\sigma^{-1} g g_\sigma \in H) = \frac{1}{\#C} \sum_{\sigma} \sum_{g \in C} I(g_\sigma^{-1} g g_\sigma \in H).$$

In order to show this is equal to $\frac{[G:H]}{\#C}\#(C \cap H)$ it is sufficient(noting that the number of σ:s is $[G : H]$ by definition) to show that $g_\sigma^{-1} g g_\sigma \in H$ iff $g \in H$.

Clearly $g_\sigma^{-1} g g_\sigma \in H$ if $g \in H$ and since the action of $g \pmod{H}$ permutes the cosets $\pmod{H}$, the reverse implication follows.

Question 13

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Question 14

We should show that $\frac{d!}{l_1! \dots l_k!} \prod_{i<j}(l_i - l_j) = \frac{d!}{\prod_{ij} h_{ij}}$, where h_{ij} are the hook lengths of the partition λ, $l_i = \lambda_i + k - i$ and $d = |\lambda|$. Note that this is the same as showing that $\prod_{i<j}(l_i - l_j) \prod_{ij} h_{ij} = l_1! \dots l_k!$.

We use induction on $|\lambda|$, observing that the statement clearly is true if $\lambda = \emptyset$.

Denote by λ' the result of removing the first column from λ. Define l', h'_{ij} et.c. in the obvious way. For $i \leq r$ we have $l'_i = l_i - 1 - (k - r)$ where r is the length of the second column of λ (or the length of the first column of λ').

We know by induction that $\prod_{1 \leq i<j \leq r}(l'_i - l'_j) \prod_{(i,j) \in \lambda'} h'_{ij} = l'_1! \dots l'_r!$. So to show that $\prod_{1 \leq i<j \leq k}(l_i - l_j) \prod_{(i,j) \in \lambda} h_{ij} = l_1! \dots l_k!$ it is enough to show that

$\prod_{1\le i\le r<j\le k}(l_i-l_j)\prod_{r\le i<j\le k}(l_i-l_j)\prod_{i=1}^{k} l_i = \prod_{1\le i\le r}\frac{l_i!}{(l_i-1-(k-r))!}\prod_{i=r+1}^{k} l_i!$, where we have used the following facts: $l_i - l_j = l'_i - l'_j$ whenever both sides are defined, $h_{i1} = l_i$ for all i, the numbers h'_{ij} and h_{ij} equal whenever both are defined (that is, whenever $j \neq 1$). By the choice of λ', we have $l_i = k - i + 1$ for $r < i \le k$. From this it follows that $\prod_{1\le i\le r<j\le k}(l_i-l_j) = \prod_{i=r+1}^{k} l_i!$ and we are left with proving that $\prod_{1\le i\le r<j\le k}(l_i-l_j)\prod_{i=1}^{k} l_i = \prod_{1\le i\le r}\frac{l_i!}{(l_i-1-(k-r))!}$. This identity is clearly true since both sides may be written as $\prod_{i=1}^{r}\prod_{j=0}^{k-r}(l_i-j)$.