The double Eulerian polynomial in terms of inversion tables

Erik Aas

Abstract

We prove a conjecture on double Eulerian polynomials due to Visontai [5], relating the number of descents and the number of inverse descents of permutations to the number of occupied rows and number of ascents of inversion tables.

Introduction

The double Eulerian polynomial $A_n(t,s)$ enumerates the number of descents and the number of inverse descents of a permutation,

$$A_n(t,s) = \sum_{\pi \in \mathbb{S}_n} t^{\mathtt{des}\,\pi} s^{\mathtt{des}\,\pi^{-1}}.$$

It is a natural generalization of the classical Eulerian polynomial $A_n(t,1)$. One interesting property of the classical Eulerian polynomial is its *unimodality*, which is an easy consequence of being nonnegative in the basis $\{t^i(1+t)^{n-i}\}_{i=0}^n$. One way of proving this is to note that $A_n(t,1)$ is the h-polynomial of the Coxeter complex (see for instance [1]). Foata and Strehl gave a bijective proof involving 'valley-hopping' - see [3] for a nice exposition. Gessel [2] conjectured that $A_n(t,s)$ similarly has nonnegative coefficients when expressed in the basis $B_{ij} = (ts)^i(t+s)^j(1+ts)^{n-2i-j}$. This conjecture has motivated some of the work on $A_n(t,s)$, including this.

For any $n \geq 0$, we denote the set of permutations of $[n] = \{1,\ldots,n\}$ (the *n-permutations*) by $\mathbb{S}_n$. We think of permutations as *words*. For example, the permutation mapping $1 \mapsto 3$, $2 \mapsto 1$, $3 \mapsto 4$, $4 \mapsto 2$ is identified with the word `3142`. An *(n-)inversion table* is a sequence $e_1,\ldots,e_n$ of integers satisfying $0 \leq e_i \leq i-1$ for each i. The set of n-inversion tables is denoted $\mathbb{I}_n$. We now define some statistics.

- $\mathtt{DES}(\pi) = \{i : \pi(i) > \pi(i+1)\}$
- $\mathtt{IDES}(\pi) = \{i : \pi^{-1}(i) > \pi^{-1}(i+1)\}$
- $\mathtt{ASC}(e) = \{i : e_i < e_{i+1}\}$ (note the strict inequality).
- $\mathtt{ROW}(e) = \{e_i : 1 \leq i \leq n\}\backslash\{0\}$.

Moreover, $\mathtt{des} = \#\mathtt{DES}, \mathtt{ides} = \#\mathtt{IDES}, \mathtt{asc} = \#\mathtt{ASC}, \mathtt{row} = \#\mathtt{ROW}$.

Examples

For example, the inversion tables of length 5 (written as words $e_1e_2e_3e_4e_5$) with 1 ascent and two occupied rows are $00221, 00211, 00210, 00021, 00032, 00031$. The permutations of length 5 with 1 descent and two inverse descents are $24135, 13524, 23514, 25134, 35124, 24513$.

We have $\texttt{DES}(24135) = \{2\}$, $\texttt{IDES}(24135) = \{1,3\}$, $\texttt{ASC}(00210) = \{2\}$, $\texttt{ROW}(00210) = \{1,2\}$.

Given a standard Young tableau P, let $\texttt{des}P = \{i : i$ is in a strictly higher row than $i+1\}$. Well-known properties for the RSK correspondence give that $A_n(t,s) = \sum_{\lambda\langle n} f_\lambda(t)f_\lambda(s)$, where $f_\lambda(x) = \sum_{P has shape \lambda} x^{\texttt{des}(P)}$. For the $n = 5$, the polynomials f_λ are, in no particular order, $1, 2x+3x^2, 4x, 2x^3+3x^2, 4x^3, x^4, 3x^2+3x$. These polynomials are easy to compute and using them we get $A_5(t,s)$.

The identity

Visontai [5] conjectured that

$$\sum_{\pi\in\mathbb{S}_n} t^{\texttt{des}\pi} s^{\texttt{ides}\pi} = \sum_{e\in\mathbb{I}_n} t^{\texttt{asc}e} s^{\texttt{row}e}.$$

In fact we will prove the following refined identity (which reduces to the conjecture by letting $t_1 = \cdots = t_n = t$). For a subset $S \subseteq [n]$, let $t^S = \prod_{i\in S} t_i$, a monomial in the indeterminates $t_1, \ldots, t_n$.

Theorem 1. *We have*

$$\sum_{\pi\in\mathbb{S}_n} t^{\texttt{DES}\pi} s^{\texttt{ides}\pi} = \sum_{e\in\mathbb{I}_n} t^{\texttt{ASC}e} s^{\texttt{row}e}.$$

The following lemma is an immediate consequence of Möbius inversion (see [4]).

Lemma 1. *Suppose α, β are functions mapping subsets of $[n]$ to polynomials. Then $\alpha = \beta$ if and only if for every $S \subseteq [n]$ we have*

$$\sum_{T\supseteq S} \alpha(T) = \sum_{T\supseteq S} \beta(T).$$

We turn to the proof of Theorem 1. Fix a subset $S \subseteq [n]$. We will prove that $\sum_{\pi\in\mathbb{S}_n:\texttt{DES}(\pi)\supseteq S} v^{\texttt{ides}(\pi)} = \sum_{e\in\mathbb{I}_n:\texttt{ASC}(e)\supseteq S} v^{\texttt{row}(e)}$.

We now construct two (rather large) rooted labelled trees.

The first tree T_1 has as node set all permutations π (of any length) such that if π is an r-permutation then $\texttt{DES}(\pi) \supseteq S \cap [r]$.

Suppose $r < s$, and that π' is an s-permutation. Denote by $\pi'_{(r)}$ the r-permutation obtained by relabeling the first r values of π' by $[r]$, retaining the relative ordering of these values. There is an edge from π to π' if $\{r, r+1, \ldots, s\} \cap S = \{r+1, \ldots, s-1\}$. All edges are given in this way. The node $\pi \in \mathbb{S}_n$ in T_1 is labelled by the pair $(\texttt{ides}(\pi), n)$

The other tree T_2 has as node set all inversion tables e (of any length) such that if e is an r-inversion table then $\texttt{ASC}(e) \supseteq S \cap [r]$.

Suppose $r < s$, e is an r-inversion table, e' is an s-inversion table. There is an edge from e to e' if the restriction of e' (considered as a function from $[r]$ to $\mathbb{Z}_{\geq 0}$) to $[r]$. The node $e \in \mathbb{I}_n$ in T_2 is labelled $(\mathtt{row}(e), n)$.

By the lemma it suffices to prove that T_1 and T_2 are isomorphic as rooted labelled trees. We prove this by constructing a bijection Φ from T_1 to T_2 inductively. Although we do not construct Φ completely explicitly, this can be done, though we have not found a particularly nice choice.

We let Φ map the root of T_1 (the empty permutation) to the root of T_2 (the empty inversion table). Now suppose π is an r-permutation and that $e = \Phi(\pi)$ has been defined, where e is an r-inversion sequence.

Let $F(\pi, s, p)$ be the set of children π' of π such that π' has length s and $\mathtt{ides}(\pi') = \mathtt{ides}(\pi) + p$. Similarly, let $G(e, s, p)$ be the set of children e' of e such that e' has length s and $\mathtt{row}(e') = \mathtt{row}(e) + p$.

To prove that Φ can be extended to the children of π it suffices to prove that $|F(\pi, s, p)| = |G(e, s, p)|$. This will finish the proof of the theorem.

It is easy to see that elements π' of $F(\pi, s, p)$ correspond in a natural way to nonnegative integer $(s+1)$-tuples $(x_1, \ldots, x_{s+1})$ with sum r such that $\sum_{i=1}^{t+1} x_i + \sum_{i=t+2}^{s+1} (x_i - 1)_+ = p$; x_i represents the number of letters in π' less than the i'th largest and the $(i+1)$'st largest among the first r letters in π'.

Similarly elements e' of $G(e, s, p)$ can naturally be represented by r-subsets S of $[r + s]$ with $|S \cap \{1, \ldots, t+1\}| = p$; S represents the values of e' on $[s]\backslash[r]$ (which are distinct by the construction of T_2).

From these remarks one easily finds $|F(\pi, s, t, p)| = \sum_{a=0}^{p} \binom{t+1}{r-p}\binom{r-a-1}{a}\binom{a+s-t-1}{a}$. Similarly, $|G(e, s, t, p)| = \binom{t+1}{r-p}\binom{r+s-t-1}{p}$. Verifying that these sums are in fact equal is routine.

References

[1] Matthias Beck and Sinai Robins: Computing the continuous discretely: Integer-point enumeration in polyhedra, Undergraduate Texts in Mathematics, Springer, New York, 2007.

[2] Petter Brändén: Actions on permutations and unimodality of descent polynomials. European J. Combin., 29(2):514-531, 2008.

[3] Kyle Petersen: Two-sided Eulerian numbers via balls in boxes. Math. Mag. (to appear). Preprint available at arXiv:1209.6273v1, 2012.

[4] Richard Stanley: Enumerative Combinatorics, Vol 2, Cambridge, 1999.

[5] Mirkó Visontai: Some remarks on the joint distribution of descents and inverse descents, Electron. J. Combin., Volume 20, Issue 1, 2013, Research article 52, 12pp.