The double Eulerian polynomial in terms of inversion tables

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Abstract

We prove a conjecture on double Eulerian polynomials due to Visontai [4], relating the number of descents and the number of inverse descents of permutations to the number of occupied rows and number of ascents of inversion tables.

The double Eulerian polynomial $A_n(t, s)$ enumerates the number of descents and the number of inverse descents of a permutation,

$$A_n(t,s) = \sum_{\pi \in \mathbb{S}_n} t^{\mathtt{des}\pi} s^{\mathtt{des}\pi^{-1}}.$$

It is a natural generalization of the classical Eulerian polynomial $A_n(t, 1)$. One interesting property of the classical Eulerian polynomial is its *unimodality*, which is an easy consequence of being nonnegative in the basis $\{t^i(1+t)^{n-i}\}_{i=0}^n$. One way of proving this is to note that $A_n(t, 1)$ is the h-polynomial of the Coxeter complex (see for instance [1]). Foata and Strehl gave a bijective proof involving 'valley-hopping' - see [3] for a nice exposition. Gessel [2] conjectured that $A_n(t, s)$ similarly has nonnegative coefficients when expressed in the basis $B_{ij} = (ts)^i(t + s)^j(1 + ts)^{n-2i-j}$. This conjecture has motivated some of the work on $A_n(t, s)$, including this.

For any $n \geq 0$, we denote the set of permutations of $[n] = \{1, \ldots, n\}$ (the *n-permutations*) by $\mathbb{S}_n$. We think of permutations as *words*. For example, the permutation mapping $1 \mapsto 3$, $2 \mapsto 1$, $3 \mapsto 4$, $4 \mapsto 2$ is identified with the word `3142`. An *(n-)inversion table* is a sequence $e_1, \ldots, e_n$ of integers satisfying $0 \leq e_i \leq i - 1$ for each i. The set of n-inversion tables is denoted $\mathbb{I}_n$. We now define some statistics.

- $\mathtt{DES}(\pi) = \{i : \pi(i) > \pi(i+1)\}$
- $\mathtt{IDES}(\pi) = \{i : \pi^{-1}(i) > \pi^{-1}(i+1)\}$
- $\mathtt{ASC}(e) = \{i : e_i < e_{i+1}\}$ (note the strict inequality).
- $\mathtt{ROW}(e) = \{e_i : 1 \leq i \leq n\} \backslash \{0\}$.

Moreover, $\mathtt{des} = \#\mathtt{DES}, \mathtt{ides} = \#\mathtt{IDES}, \mathtt{asc} = \#\mathtt{ASC}, \mathtt{row} = \#\mathtt{ROW}$.

Visontai [4] conjectured that

$$\sum_{\pi \in \mathbb{S}_n} t^{\mathtt{des}\pi} s^{\mathtt{ides}\pi} = \sum_{e \in \mathbb{I}_n} t^{\mathtt{asc}e} s^{\mathtt{row}e}$$

for all n.

Examples

For example, the inversion tables of length 5 (written as words $e_1e_2e_3e_4e_5$) with 1 ascent and two occupied rows are $00221, 00211, 00210, 00021, 00032, 00031$. The permutations of length 5 with 1 descent and two inverse descents are $24135, 13524, 23514, 25134, 35124, 24513$.

We have $\texttt{DES}(24135) = \{2\}$, $\texttt{IDES}(24135) = \{1, 3\}$, $\texttt{ASC}(00210) = \{2\}$, $\texttt{ROW}(00210) = \{1, 2\}$.

References

[1] Matthias Beck and Sinai Robins: Computing the continuous discretely: Integer-point enumeration in polyhedra, Undergraduate Texts in Mathematics, Springer, New York, 2007.

[2] Petter Brändén: Actions on permutations and unimodality of descent polynomials. European J. Combin., 29(2):514-531, 2008.

[3] Kyle Petersen: Two-sided Eulerian numbers via balls in boxes. Math. Mag. (to appear). Preprint available at arXiv:1209.6273v1, 2012.

[4] Mirkó Visontai: Some remarks on the joint distribution of descents and inverse descents, Electron. J. Combin., Volume 20, Issue 1, 2013, Research article 52, 12pp.